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DETERMINATION OF THE SOURCE IN QUASILINEAR EQUATIONS OF THE PARABOLIC TYPE

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The uniqueness theorem is proven for the solution of the two-dimensional inverse problem for an unknown source function dependent on the solution of the direct problem and on the spatial coordinate.

We consider the heat equation in the region $D(T, x_0) = \{x_0 < x < \infty, 0 < t \le T\}$

$$u_t = u_{xx} + q(u, x) + f(x, t) \tag{1}$$

subject to the boundary and initial conditions

$$u|_{t=0} = 0, \ x_0 \leqslant x < \infty, \tag{2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_0} = 0, \ 0 \leqslant t \leqslant T, \tag{3}$$

plus the following condition on the solution at the point $x = x_0$:

$$u|_{x=x_0} = \psi(t, x_0), \ 0 \leqslant t \leqslant T. \tag{4}$$

We assume that the parameter x_0 could range from zero to infinity. The problem is to determine the function $q(u, x)$ for a given function $\psi(t, x_0)$.

THEOREM. Let the functions $f(x, t)$ and $\psi(t, x_0)$ satisfy the conditions $f(x, t) \in C[0, \infty) \times [0, T])$, $\psi(t, x_0) \in C^{1,0}([0, T] \times [0, \infty))$ and $\psi_t'(t, x_0) \ge \gamma$ for $t \in [0, T]$, where γ is a sufficiently large positive number. In addition, the consistency condition $\psi(0, x_0) = 0$ is assumed to be satisfied.

Then the solution $q(u, x)$ of the inverse problem is unique in the class of functions $q(u, x) \in C^{1,1}((-\infty, \infty) \times [0, \infty))$ satisfying the conditions

$$-q(0, x) \leqslant f(x, t) \leqslant \psi_t'(t, x_0) - q(\psi(t, x_0), x), \ x \in [x_0, \infty), \ t \in [0, T],$$

and for two arbitrary functions of this class $q_1(u, x)$ and $q_2(u, x)$ their difference $q(u, x) = q_1(u, x) - q_2(u, x)$ satisfies the inequality

$$\|q(u, x)\|_{W_\infty^{0,\alpha}([R_1, R_2] \times [0, \infty))} \leqslant c \|q(u, x)\|_{L_\infty([R_1, R_2] \times [0, \infty))}$$

for a certain value of $\alpha \in (0, 1)$.

Proof. The following maximum principle holds for the assumptions of the theorem.

Maximum Principle. If the conditions of the theorem are satisfied, then $u(x, t) \in C^{2,1}((0, \infty) \times (0, T]) \cap C([0, \infty) \times [0, T]) \cap L_\infty((0, \infty) \times (0, T])$ and the solution of the problem (1), (2), (4) satisfies the condition $0 \le u(x, t) \le \psi(t, x_0)$, $(x, t) \in D(T, x_0)$.

Furthermore suppose that there exist two solutions of the problem (1)-(4): $\{u_1(x, t), q_1(u_1, x)\}$ and $\{u_2(x, t), q_2(u_2, x)\}$. Then, putting $x = x_0$ in (1), we obtain the relation

$$q(\psi(t, x_0), x_0) = \psi_t'(t, x_0) - f(x_0, t) - u_{xx}|_{x=x_0}, \tag{5}$$

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from which it follows that

$$(q_1 - q_2)(\psi(t, x_0), x_0) = -(u_{1xx} - u_{2xx})|_{x=x_0}. \tag{6}$$

We rewrite the problem (1)-(3) in terms of the new variable $y = x - x_0$:

$$u_t = u_{yy} + q(u, y + x_0) + f(y + x_0, t), \ (y, t) \in D(T, 0), \tag{7}$$

$$u|_{t=0} = 0, \ 0 \leqslant y < \infty, \tag{8}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \ 0 \leqslant t \leqslant T. \tag{9}$$

The solution of the boundary-value problem (7)-(9) can be represented in the integral form

$$u(y, t) = \int_0^t \int_0^\infty \frac{1}{2\sqrt{\pi(t-\tau)}} \left[\exp\left\{-\frac{(y-\xi)^2}{4(t-\tau)}\right\} + \exp\left\{-\frac{(y+\xi)^2}{4(t-\tau)}\right\}\right] \{q(u, \xi + x_0) + f(\xi + x_0, \tau)\} \, d\xi d\tau.$$

Using this representation, (6) can be rewritten as

$$(q_1 - q_2)(\psi(t, x_0), x_0) = -2 \int_0^t \int_0^\infty \frac{\partial}{\partial t} \left[\frac{1}{2\sqrt{\pi(t-\tau)}} \times \exp\left\{-\frac{\xi^2}{4(t-\tau)}\right\}\right] \{q_1(u_1, \xi + x_0) - q_2(u_2, \xi + x_0)\} \, d\xi d\tau,$$

or in terms of the new variable of integration $\xi + x_0 = \eta$:

$$(q_1 - q_2)(\psi(t, x_0), x_0) = -2 \int_0^t \int_{x_0}^\infty \frac{\partial}{\partial t} \left[\frac{1}{2\sqrt{\pi(t-\tau)}} \times \exp\left\{-\frac{(\eta - x_0)^2}{4(t-\tau)}\right\}\right] \{q_1(u_1, \eta) - q_2(u_2, \eta)\} \, d\eta d\tau. \tag{10}$$

We denote

$$G(y, \xi, t, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \left[\exp\left\{-\frac{(y-\xi)^2}{4(t-\tau)}\right\} + \exp\left\{-\frac{(y+\xi)^2}{4(t-\tau)}\right\}\right].$$

Then

$$\frac{\partial}{\partial y} \int_0^\infty G(y, \xi, t, \tau) \, dy = 0.$$

Hence (10) can be rewritten in the form

$$(q_1 - q_2)(\psi(t, x_0), x_0) = -\int_0^t \int_{x_0}^\infty \frac{\partial}{\partial t} \left[\frac{1}{\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{(\eta - x_0)^2}{4(t-\tau)}\right\}\right] \times$$
$$\times \{(q_1 - q_2)(u_1, \eta) - (q_1 - q_2)(\psi(\tau, x_0), x_0) + (q_2(u_1, \eta) - q_2(u_2, \eta))\} \, d\eta d\tau. \tag{11}$$

We obtain from (11)

$$|(q_1 - q_2)(\psi(t, x_0), x_0)|_{L_\infty(D(t,0))} \leqslant c \int_0^t \frac{1}{(t-\tau)^{3/2}} \int_{x_0}^\infty \exp\left\{-\frac{(\eta - x_0)^2}{4(t-\tau)}\right\} \times$$
$$\times \{|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(\tau,0))} (|u_1 - \psi(\tau, x_0)|^\alpha + (\eta - x_0)^{2\alpha}) + |q_2|_{W_\infty^{1,0}(D(\tau,0))} |u_1(\eta, \tau) - u_2(\eta, \tau)|\} \, d\eta d\tau. \tag{12}$$

Introducing the norm

$$\|u\|_{L_\infty^\beta(D(t,0))} = \operatorname{ess} \sup_{(x,t) \in D(t,0)} \exp\{-\beta t\} |u(x, t)|$$

and choosing the number β appropriately, it can be shown that $\|u_1 - u_2\|_{L_\infty(D(t,0))}$ has the bound

$$\|u_1 - u_2\|_{L_\infty(D(t,0))} \leqslant c \|q_1 - q_2\|_{L_\infty(D(t,0))}.$$

Representing $\|u_1 - u_2\|_{W_\infty^{\alpha, 2\alpha}(D(t,0))}$ in integral form, we obtain after some simple reductions

$$\|u_1 - u_2\|_{W_\infty^{1, 1/2}(D(t,0))} \leqslant c \|q_1 - q_2\|_{L_\infty(D(t,0))}. \tag{13}$$

We proceed to bound the right-hand side of the inequality (12):

$$|(q_1 - q_2)(\psi(t, x_0), x_0)|_{L_\infty(D(t,0))} \leqslant c \left| \int_0^t \frac{1}{(t-\tau)^{3/2}} \int_{x_0}^\infty \exp\left\{-\frac{(\eta - x_0)^2}{4(t-\tau)}\right\} \times \right.$$
$$\times \{|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(\tau,0))} ([u_1 - \psi(\tau, x_0)]^\alpha + (\eta - x_0)^{2\alpha}) +$$
$$\left. + (\eta - x_0) \|q_1 - q_2\|_{L_\infty(D(\tau,0))}\} \, d\eta d\tau \right| . \tag{14}$$

We formulate the following lemma in order to further bound the right-hand side of the inequality (14):

<u>LEMMA</u>. Let

$$Nu_1(\eta, \tau) = \frac{1}{(\eta - x_0)^2} [u_1(\eta, \tau) - \psi(\tau, x_0)],$$

then $|Nu_1(\eta, \tau)| \leq c\|f(x, t)\|_{W_\infty^{1,0}(D(t,0))}$. The proof of this result is not complicated and has been given by Lorentz.

Taking into account this lemma, (14) takes the form

$$|(q_1 - q_2)(\psi(t, x_0), x_0)|_{L_\infty(D(t,0))} \leqslant c \int_0^t \frac{1}{(t-\tau)^{3/2}} \int_{x_0}^\infty \exp\left\{-\frac{(\eta - x_0)^2}{4(t-\tau)}\right\} \times$$
$$\times \{(\eta - x_0)^{2\alpha} |q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(\tau,0))} + (\eta - x_0) \|q_1 - q_2\|_{L_\infty(D(\tau,0))}\} \, d\eta d\tau.$$

To obtain the final bound, from which the uniqueness of the solution of the inverse problem will follow, it remains to bound $|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(t,0))}$. Assuming arbitrarily that $0 < t_1 < t_2 < T$, $0 < x_0^1 < x_0^2 < \infty$ and using (11), we obtain

$$|q_1(\psi(t_2, x_0^2), x_0^2) - q_2(\psi(t_2, x_0^2), x_0^2) - q_1(\psi(t_1, x_0^1), x_0^1) +$$
$$+ q_2(\psi(t_1, x_0^1), x_0^1)| \leqslant c \{[(x_0^2 - x_0^1)^{2\alpha} + (t_2 - t_1)^\alpha] [|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(\psi(t, x_0), 0)} + |q_1 - q_2|_{L_\infty(D(t,0))}]\}.$$

We then obtain the inequality

$$|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(t,0))} \leqslant c (|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(\psi(t, x_0), 0))} + |q_1 - q_2|_{L_\infty(D(t,0))}).$$

From the condition imposed on the function $\psi(t, x_0)$ in the statement of the theorem, namely that $\psi_t'(t, x_0) \geq \gamma$, we have

$$|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(t,0))} \geqslant \gamma^\alpha |q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(\psi(t, x_0), 0)}.$$

Using this result, we obtain

$$|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(\psi(t, x_0), 0))} \leqslant (\gamma^\alpha - c) c |q_1 - q_2|_{L_\infty(D(t,0))}.$$

For sufficiently large γ we have

$$|q_1 - q_2|_{W_\infty^{\alpha, 2\alpha}(D(\psi(t, x_0), 0)} \leqslant c |q_1 - q_2|_{L_\infty(D(t,0))}.$$

Finally we obtain

$$|q_1 - q_2|_{L_\infty(D(t,0))} \leqslant c \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} |q_1 - q_2|_{L_\infty(D(\tau,0))} \, d\tau.$$

This is inequality of Volterra type and it therefore follows that the solution of the inverse problem is unique.

NOTATION

$D(T, x_0)$, domain of the solution of the direct problem; $C([0, \infty) \times [0, T])$, space of the continuous functions; $C^{\alpha,\beta}([0, \infty) \times [0, T])$, space of the continuous functions with derivative of order α with respect to the spatial variable and with derivative of order β with respect to time; $\int_a^b$, definite integral; exp, exponential function; ess sup, essential supremum; x, spatial variable; t, time; ∞, infinity; c, positive constant; π, number approximately equal to 3.14; $\partial/\partial t$, $\partial/\partial x$, partial derivatives with respect to time and space, respectively.

DETERMINATION OF THE CONTACT THERMAL RESISTANCE FROM THE SOLUTION OF THE INVERSE PROBLEM OF THERMAL CONDUCTIVITY

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The contact resistance at the boundary between an orthropic reinforcing rod and an isotropic matrix is determined from the solution of the inverse problem of thermal conductivity, using the gradient method. The suggested modification of the computational algorithm as the initial calculation of the initial period of the thermal process is shown to enhance the resolving power of the method and the choice of zeroth approximations from below is shown to ensure monotonic convergence of the solution.

One parameter which determines the heat exchange in reinforced materials or in elements of complex structures is the contact thermal resistance (CTR) due to the nonideal mechanical coupling of the contact surfaces. In theoretical studies on CTR the contribution of the thermal resistance to the heat transfer across the contact interface of the media is described by a condition in the form

$$\lambda_1 \frac{\partial T_1}{\partial n} = \lambda_2 \frac{\partial T_2}{\partial n}, \quad -\lambda_1 \frac{\partial T_1}{\partial n} R = T_2 - T_1,$$

where R is the contact thermal resistance, λ_1 and λ_2 are the thermal conductivities of the media in contact, and n is the normal to the contact surface. Thermal contact resistance has been considered as a function of the determining parameters, e.g., temperature [1], thermal stresses [2], and a complex of parameters in the form of the compression pressure, the instantaneous tensile strength, and the height of the irregularities [3]. Nevertheless, even though different determining parameters are chosen, the value of the thermal resistance for each specific case is determined experimentally or is approximated [4].

Artyukhin and Nenarokomov [5] advanced a fairly effective treatment for determining CTR as a function of the temperature on the basis of the solution of the inverse one-dimensional problem. In view of this, it is of some interest to extrapolate this treatment to the two-dimensional case and to study the possibilities of an algorithm for the solution.

The CTR is reconstructed on the example of an orthotropic cylindrical region surrounded by an isotropic medium. The mathematical simulation of the heat transfer in the media in contact was presented in the form of a two-dimensional, nonstationary system of equations involving the temperature dependence of the coefficient being sought:

$$\frac{\partial T_1}{\partial t} = a_1 \left(\frac{\partial^2 T_1}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_1}{\partial r} \right) \right); \tag{1}$$

$$\frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2}{\partial z^2} + a_3 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_2}{\partial r} \right);$$

$$T_1(z, r, 0) = T_2(z, r, 0) = T_0 = \text{const}; \tag{2}$$

$$\frac{\partial T_1(0, r, t)}{\partial z} = \frac{\partial T_2(0, r, t)}{\partial z} = 0;$$

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